# A frequency domain approach for blind identification with filter bank precoders

P. P. Vaidyanathan and Bojan Vrcelj
Dept. Electrical Engineering, California Institute of Technology
Pasadena, CA 91125, USA
ppvnath@systems.caltech.edu, bojan@systems.caltech.edu

***Abstract.***[1] It is well-known that filter bank precoders can be used for blind identification as well as equalization of FIR channels. In this paper we introduce a new blind identification scheme which directly identifies the frequency domain equalizer coefficients. The precoder redundancy required for this is the same as in the earlier methods, but the proposed method offers simplicity. For example closed form formulas are involved rather than iterative computation of annihilating eigenvectors as in earlier methods.

## I. INTRODUCTION

Figure 1(a) shows a digital transmultiplexer structure used in communications. In recent years this structure has been studied in great depth [1]-[4]. Its usefulness in channel equalization and blind identification has been recognized [1], [4]. A tutorial overview of the theory and applications of this system is available in the companion paper [6]. In the system shown we can regard $s_k(n)$ as symbol streams from $M$ users. In some applications these independent streams may have been derived from a single user (as in DMT systems) but this detail is not relevant in our discussion here. In general the received signal $\widehat{s}_k(n)$ suffers from interference from other users $(s_m(n), m \neq k)$ and also from distortion due to the noisy channel $C(z)$. We will assume that the channel is FIR with order $\leq L$,

$$C(z) = \sum_{n=0}^{L} c(n) z^{-n} \tag{1}$$

We also assume $P > M$, so the transmultiplexer has redundancy. More specifically, we let

$$P = M + L \tag{2}$$

as in [2], [3]. Writing the filters in polyphase form [5]

$$H_m(z) = \sum_{k=0}^{P-1} z^k E_{mk}(z^P), \; F_m(z) = \sum_{k=0}^{P-1} z^{-k} R_{km}(z^P) \tag{3}$$

we can redraw Fig. 1(a) as in Fig. 1(b). The system shown in the box is the blocked version $\mathbf{C}_b(z)$ of the channel. As in [2]–[4] we constrain $\mathbf{E}(z)$ and $\mathbf{R}(z)$ to be constants $\mathbf{E}$ and $\mathbf{R}$. Then the filters $F_k(z)$ and $H_k(z)$ have order $\leq P-1$. We further assume as in [3] that

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{0} \end{bmatrix}$$

which is called the *zero-padding* constraint. Here $\mathbf{R}_1$ is $M \times M$, and is referred to as the **precoder** matrix.

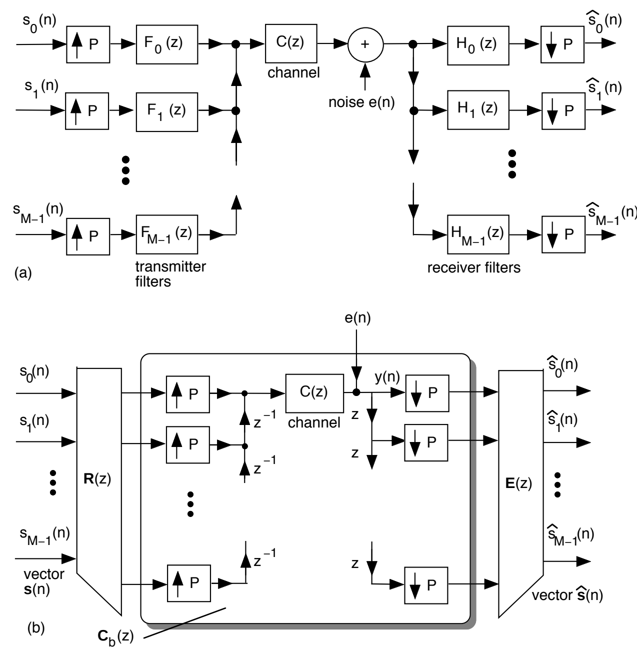


**Fig. 1**. (a) The $M$-user transmultiplexer, and (b) polyphase version.

With filters restricted as above, we have (ignoring noise)

$$\widehat{\mathbf{s}}(n) = \mathbf{E}\mathbf{A}\mathbf{R}_1 \mathbf{s}(n) \tag{4}$$

$\mathbf{A}$ represents the effect of the channel completely:

$$\mathbf{A} = \begin{bmatrix} c(0) & 0 & \ldots & 0 \\ c(1) & c(0) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c(L) & & & \\ 0 & c(L) & & \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & c(L) \end{bmatrix} \tag{5}$$

***Aim of the paper.*** If the FIR channel $C(z)$ is known, then by appropriate choice of FIR filters $F_k(z)$ and $H_k(z)$,

[1] Work supported in part by the ONR grant N00014-99-1-1002, USA.

interuser interference and channel distortion can be completely eliminated [2],[3],[6], so that $\widehat{s}_k(n) = s_k(n)$ in absence of noise. Furthermore, when the channel coefficients $c(n)$ are unknown the redundancy (2) also allows the coefficients $c(n)$ to be identified from finite measurements of the received signal $y(n)$ without knowledge of $s_k(n)$. This blind identification method was developed in [4]. In Sec. II we briefly review this. We then introduce in Sec. III a new method for blind identification which we call the *frequency domain approach.* The main advantage of the method is conceptual and practical simplicity: for example closed form formulas are involved rather than iterative computation of annihilating eigenvectors. Moreover, since Eq. (2) is still used, the redundancy required is the same as in earlier methods. Simulation results and conclusions are presented in Sec. IV.

## II. BLIND IDENTIFICATION BASICS

Figure 2 shows the path from the transmitted symbols to the channel output $y(n)$. For convenience we consider the blocked version $\mathbf{y}(n)$ as indicated. With the assumptions described at the beginning of Sec. I we have $\mathbf{y}(n) = \mathbf{AR}_1\mathbf{s}(n)$ (ignoring noise). Assuming the channel $c(n)$ is unknown, here is how we can identify it upto scale: imagine we observe the output vector $\mathbf{y}(n)$ for a certain duration, say $0 \le n \le J-1$, and write the equation

$$\underbrace{\begin{bmatrix}\mathbf{y}(0) & \mathbf{y}(1) & \dots & \mathbf{y}(J-1)\end{bmatrix}}_{\mathbf{Y}\text{ matrix; size } P\times J}$$

$$= \underbrace{\mathbf{A}}_{P\times M}\ \underbrace{\mathbf{R}_1}_{M\times M}\ \underbrace{\begin{bmatrix}\mathbf{s}(0) & \mathbf{s}(1) & \dots & \mathbf{s}(J-1)\end{bmatrix}}_{\mathbf{S}\text{ matrix; size } M\times J} \qquad (6)$$

At this point we assume that the symbol stream $\mathbf{s}(n)$ is *rich,* that is, there exists a $J$ such that $\mathbf{S}$ has full rank $M$. Since $\mathbf{A}$ and $\mathbf{R}_1$ have rank $M$, the product on the right hand side of Eq. (6) has rank $M$. So the $P \times J$ data matrix $\mathbf{Y}$ has rank $M$, and there are $P - M$ or $L$ linearly independent vectors orthogonal to all the columns in $\mathbf{Y}$. That is, there is a $L \times P$ matrix $\mathbf{V}$ with $L$ independent rows such that

$$\mathbf{VY} = \mathbf{VAR}_1\mathbf{S} = \mathbf{0} \qquad (7)$$

Since $\mathbf{R}_1\mathbf{S}$ has rank $M$, this implies

$$\mathbf{VA} = \mathbf{0} \qquad (8)$$

As $\mathbf{V}$ is $L \times P$ with rank $L$, there are $P - L = M$ independent *columns* which annihilate $\mathbf{V}$ from the right. But the $M$ columns of the lower triangular matrix $\mathbf{A}$ are linearly independent and annihilate $\mathbf{V}$, so *any annihilator* of $\mathbf{V}$ is in the column space of $\mathbf{A}$. In particular consider nonzero vectors of the form $\begin{pmatrix}\times \\ \mathbf{0}\end{pmatrix}$ where $\times$ has length $L+1$. The only vector of this form which annihilates $\mathbf{V}$ from the right is the 0th column of $\mathbf{A}$. This column (hence $c(n)$) can therefore be identified upto scale. A clever variation of this method for the case of noisy channels is also described in [4], and the basic principle is similar.

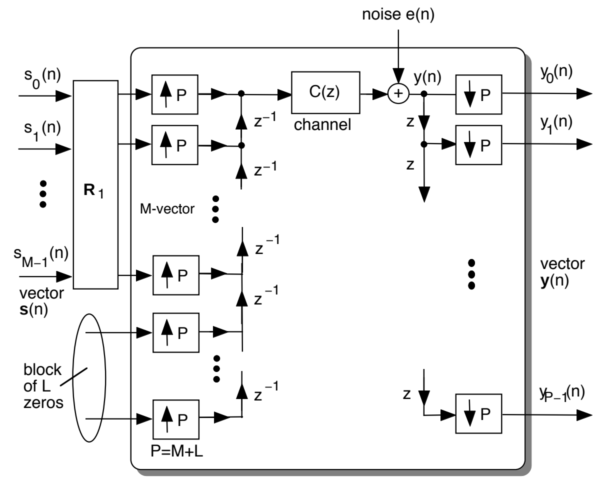


**Fig. 2**. The zero-padding system with precoder $\mathbf{R}_1$.

## III. THE FREQUENCY DOMAIN APPROACH

It is possible to choose the filters in the zero-padded transceiver in such a way that samples of the channel $z$-transform $C(z)$ at $M$ distinct points $z = \rho_k$ appear in the equalization equations at the receiver. A familiar example is the DMT system where $\rho_k = e^{j2\pi k/M}$ (though the DMT system uses cyclic-prefixing rather than zero padding). We will start from a zero-padded system and design it such that the multipliers $1/C(\rho_k)$ appear as equalizers. We then show how to perform blind identification of $C(\rho_k)$ directly without going through time domain computation of annihilating eigenvectors. The procedure will therefore turn out to be quite simple.

The matrix $\mathbf{A}$ in Eq. (5) is a $P \times M$ *full-banded Toeplitz matrix*, and therefore satisfies the identity

$$\begin{bmatrix}1 & \rho_k^{-1} & \dots & \rho_k^{-(P-1)}\end{bmatrix}\mathbf{A}$$
$$= C(\rho_k)\begin{bmatrix}1 & \rho_k^{-1} & \dots & \rho_k^{-(M-1)}\end{bmatrix} \qquad (9)$$

for any $\rho_k$. We will exploit this. First we choose the receiver filters as

$$H_k(z) = \frac{1}{C(\rho_k)}\Big(1 + \rho_k^{-1}z + \ldots + \rho_k^{-(P-1)}z^{P-1}\Big) \quad (10)$$

where $\rho_k$ are distinct for $0 \le k \le M-1$. The polyphase matrix of the receiver filters is a $M \times P$ matrix:

$$\mathbf{E}(z) = \mathbf{\Lambda}_M^{-1}\underbrace{\begin{bmatrix}1 & \rho_0^{-1} & \dots & \rho_0^{-(P-1)} \\ 1 & \rho_1^{-1} & \dots & \rho_1^{-(P-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \rho_{M-1}^{-1} & \dots & \rho_{M-1}^{-(P-1)}\end{bmatrix}}_{\text{Vandermonde matrix } V_{M\times P}} \qquad (11)$$

where $\mathbf{\Lambda}_M$ is a $M \times M$ diagonal matrix with diagonal elements $C(\rho_0), \ldots, C(\rho_{M-1})$. From Eq. (4) we know $\widehat{\mathbf{s}}(n) = \mathbf{EAR}_1\mathbf{s}(n)$, ignoring noise. Letting $\mathbf{V}_{M\times M}$ denote the leftmost $M \times M$ submatrix of the Vandermonde matrix in Eq. (11), and using property property (9), we have

$$\widehat{\mathbf{s}}(n) = \mathbf{\Lambda}_M^{-1}\mathbf{\Lambda}_M \mathbf{V}_{M\times M}\mathbf{R}_1\mathbf{s}(n).$$

By choosing $\mathbf{R}_1 = \mathbf{V}_{M\times M}^{-1}$ we therefore have $\mathbf{EAR}_1 = \mathbf{I}_M$, ensuring *perfect symbol recovery* in absence of noise.

### *III.1. Introduction of matrix* $\mathbf{G}$

Having defined the structure of the transceiver filter system, we now show how to identify $C(\rho_k)$ from the channel output. For this we introduce a matrix $\mathbf{G}$ as shown in Fig. 3. (*Note*: $\mathbf{G} = \mathbf{I}$ in Fig. 2).

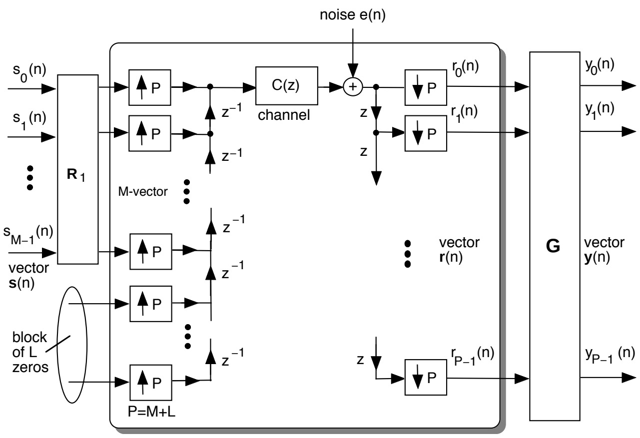


**Fig. 3**. The modified structure for discussion of blind identification.

The matrix $\mathbf{G}$ is chosen as a natural extension of the polyphase matrix $\mathbf{E}(z)$, by adding $L = P - M$ rows:[2]

$$\mathbf{G} = \begin{bmatrix} 1 & \rho_0^{-1} & \cdots & \rho_0^{-(P-1)} \\ 1 & \rho_1^{-1} & \cdots & \rho_1^{-(P-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \rho_{M-1}^{-1} & \cdots & \rho_{M-1}^{-(P-1)} \\ ---- & ---- & ---- & ---- \\ 1 & \rho_M^{-1} & \cdots & \rho_M^{-(P-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \rho_{P-1}^{-1} & \cdots & \rho_{P-1}^{-(P-1)} \end{bmatrix}$$

Here $\rho_0, \rho_1, \ldots \rho_{P-1}$ are distinct nonzero numbers. $\mathbf{G}$ is a Vandermonde matrix, and in our notation it would be $\mathbf{V}_{P\times P}$. The received signal $\mathbf{y}(n)$ can be written as

$$\mathbf{y}(n) = \mathbf{GAR}_1\mathbf{s}(n)$$

in absence of noise. In view of the identity (9) we have

$$\mathbf{y}(n) = \begin{bmatrix} \mathbf{\Lambda}_M & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_L \end{bmatrix} \begin{bmatrix} \mathbf{V}_{M\times M} \\ \mathbf{V}_{L\times M} \end{bmatrix} \mathbf{R}_1\mathbf{s}(n)$$

where $\mathbf{\Lambda}_L$ is a $L \times L$ diagonal matrix with diagonal elements $C(\rho_M), \ldots, C(\rho_{P-1})$, and $\mathbf{V}_{L\times M}$ is an $L \times M$ Vandermonde matrix obtained by retaining the last $L$ rows and first $M$ columns of $\mathbf{G}$. Since $\mathbf{V}_{M\times M}\mathbf{R}_1 = \mathbf{I}_M$, this becomes

$$\mathbf{y}(n) = \begin{bmatrix} \mathbf{\Lambda}_M & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_L \end{bmatrix} \begin{bmatrix} \mathbf{I}_M \\ \mathbf{B} \end{bmatrix} \mathbf{s}(n)$$

where $\mathbf{B} = \mathbf{V}_{L\times M}\mathbf{R}_1$ is an $L \times M$ matrix.

### *III.2. Annihilators of the data matrix* $\mathbf{Y}$

Now assume that we have accumulated the output vectors $\mathbf{y}(0), \mathbf{y}(1)\ldots, \mathbf{y}(J-1)$ to obtain the matrix $\mathbf{Y}$ as before:

$$\mathbf{Y} = \underbrace{\begin{bmatrix} \mathbf{\Lambda}_M & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_L \end{bmatrix}}_{P\times P} \underbrace{\begin{bmatrix} \mathbf{I}_M \\ \mathbf{B} \end{bmatrix}}_{P\times M} \underbrace{\mathbf{S}}_{M\times J} \tag{12}$$

where $\mathbf{S}$ is $M \times J$ and $J$ is large enough so that $\mathbf{S}$ has rank $M$ (richness assumption). Assuming $C(\rho_k) \neq 0, \quad 0 \leq k \leq P-1$, the above product $\mathbf{Y}$ has rank $M$. This means that $\mathbf{Y}$ has $L$ left-annihilators. Since $\mathbf{S}$ has rank $M$, these annihilators are also the annihilators of the matrix

$$\mathbf{C} \triangleq \begin{bmatrix} \mathbf{\Lambda}_M & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_L \end{bmatrix} \begin{bmatrix} \mathbf{I}_M \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{\Lambda}_M \\ \mathbf{\Lambda}_L\mathbf{B} \end{bmatrix} \tag{13}$$

Observe now that

$$[\, \mathbf{\Lambda}_L\mathbf{B}\mathbf{\Lambda}_M^{-1} \quad -\mathbf{I}_L \,] \begin{bmatrix} \mathbf{\Lambda}_M \\ \mathbf{\Lambda}_L\mathbf{B} \end{bmatrix} = \mathbf{0}$$

which shows that all the $L$ rows of the matrix on the left are precisely the left-annihilators of $\mathbf{C}$. These are also the left annihilators of $\mathbf{Y}$. In practice we can identify these annihilators by looking for annihilators of $\mathbf{Y}$ of the form

$$\mathbf{v}_k^\dagger = [\times \times \ldots \times 0 \ldots 010 \ldots 0], \quad 0 \leq k \leq L-1.$$

There are $M$ elements indicated as $\times$ (to be determined), and the 1 occurs at the $k$th place following the $\times$'s. We see that the data matrix $\mathbf{Y}$ has a $L \times P$ left-annihilator of the form $[\, \mathbf{Z}_0 \quad -\mathbf{I}_L \,]$ so that

$$L \; (\overset{M}{\mathbf{Z}_0} \quad \overset{L}{-\mathbf{I}_L}) \underbrace{\begin{bmatrix} \mathbf{Y}_0 \\ \mathbf{Y}_1 \end{bmatrix}}_{\mathbf{Y}} = \mathbf{0}$$

So there exists a $\mathbf{Z}_0$ such that $\mathbf{Z}_0\mathbf{Y}_0 = \mathbf{Y}_1$ or

$$\mathbf{Z}_0 = \mathbf{Y}_1\mathbf{Y}_0^\dagger[\mathbf{Y}_0\mathbf{Y}_0^\dagger]^{-1} \tag{14}$$

Notice from (12) that $\mathbf{Y}_0 = \mathbf{\Lambda}_M\mathbf{S}$, which therefore has rank $M$ (because of the assumptions that $\mathbf{\Lambda}_M$ and $\mathbf{S}$ have rank $M$), so the inverse indicated in Eq. (14) exists. Summarizing, the $L$ left-annihilators of the data matrix $\mathbf{Y}$ can be obtained in essentially a closed form formula! Thus the annihilators are the $L$ rows of the matrix[3]

$$\mathbf{V} \triangleq L \; (\overset{M}{\mathbf{Y}_1\mathbf{Y}_0^\dagger[\mathbf{Y}_0\mathbf{Y}_0^\dagger]^{-1}} \quad \overset{L}{-\mathbf{I}_L}) \tag{15}$$

[2] Once the channel is identified, $\mathbf{G}$ is replaced, e.g., with $\mathbf{E}$ to reconstruct the symbols (zero-forcing solution).

[3] The reason why the annihilators turn out to be unique is because we are looking for annihilators of a restricted form.

*III.3. Obtaining the channel from annihilators of data* $\mathbf{Y}$

Observe that the matrix $\mathbf{V}$ which annihilates the data $\mathbf{Y}$ also annihilates the matrix $\mathbf{C}$ so that

$$[\,\mathbf{Z}_0 \quad -\mathbf{I}_L\,]\underbrace{\begin{bmatrix}\mathbf{\Lambda}_M\\ \mathbf{\Lambda}_L\mathbf{B}\end{bmatrix}}_{\mathbf{C}} = \mathbf{0} \qquad (16)$$

The matrices $\mathbf{Z}_0$ and $\mathbf{B}$ are known. Since $\mathbf{\Lambda}_M$ and $\mathbf{I}_L$ are diagonal matrices, we can obtain an estimate of the channel practically *by inspection of any single row* of the preceding equation!

For example denote the $k$th row elements of $\mathbf{Z}_0$ by $z_{km}$ and the $k$th row of $\mathbf{B}$ by $\mathbf{B}_k$. We have from (16)

$$[z_{k0}C(\rho_0), z_{k1}C(\rho_1), \ldots, z_{k,M-1}C(\rho_{M-1})] = C(\rho_{M+k})\mathbf{B}_k \qquad (17)$$

From this, the elements $C(\rho_k),\ 0 \leq k \leq M-1$ arising in equalization can be readily estimated upto scale. This estimation, repeated for $0 \leq k \leq L-1$, does not give identical results when there is noise. So this calculation should be repeated for all $L$ rows of (17), and averaged. This yields an answer robust to channel noise.

The channel estimation process is now complete, but we can further reduce the effect of noise as follows: By considering the $k$th column of Eq. (16) (instead of $k$th row), we can obtain an estimate of the $L+1$ numbers $C(\rho_k), C(\rho_M), C(\rho_{M+1}), \ldots, C(\rho_{M+L-1})$ upto scale. This can be used to estimate the $L+1$ time-domain coefficients $c(n)$ upto scale. This estimation process can be repeated for each of the $M$ columns, and averaged to reduce the effect of noise. From this estimate of $c(n)$, the quantities $C(\rho_k), 0 \leq k \leq M-1$ required in equalization can then be identified. The two estimates of $C(\rho_k), 0 \leq k \leq M-1$ (one from considering the rows and the other considering the columns of (16)) can finally be averaged.

*III.4. Frequency domain equalizers*

The quantities $\rho_k$ can be chosen to have unit magnitude, so that $C(\rho_k)$ are samples of the channel frequency response $C(z)$. An example would be $\rho_k = e^{j2\pi k/M}$ for $0 \leq k \leq M-1$ (as in DMT systems) and $\rho_k = e^{j2\pi(k-M+0.5)/M}$ for $M \leq k \leq P-1$. With $L < M$ as typically is the case, the quantities $C(\rho_k)$ are samples of the channel frequency response at distinct frequencies. This is demonstrated in Fig. 4 for $M = 10$ and $P = 14$, where the heavy samples are $C(\rho_M), \ldots, C(\rho_{P-1})$.

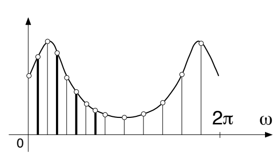


**Fig. 4**. Samples of the channel Fourier transform $C(e^{j\omega})$ for the case where $M = 10$ and $P = 14$.

Since the method described in the preceding pages identifies $C(\rho_0), \ldots, C(\rho_{M-1})$ directly, we refer to it as the *frequency domain approach.*

## IV. CONCLUDING REMARKS AND EXAMPLES

To demonstrate the idea with an example, we consider a simple 4th order FIR channel ($L = 4$) with $C(z)$ given by

$$-0.7684 - 0.8655z^{-1} + 0.4305z^{-2} - 0.3204z^{-3} + 0.4992z^{-4}$$

We choose a single user sending a 64-QAM symbols, and assume that $\mathbf{s}(n)$ is a blocked version with $M = 12$ so there are 12 subusers $s_k(n)$, and $P = M + L = 16$. Assuming the noise $e(n)$ is white and the SNR at the channel output is 25 dB, we estimate the channel using the traditional method (Sec. II) and the new frequency domain method (Sec. III). For the traditional method the precoder $\mathbf{R}_1$ was the IDFT matrix. For the frequency domain method, $\rho_k$ were as in Sec. III.4. Once the channel is estimated, it can be used in equalization. The scatter diagrams of the equalized symbols are shown below.

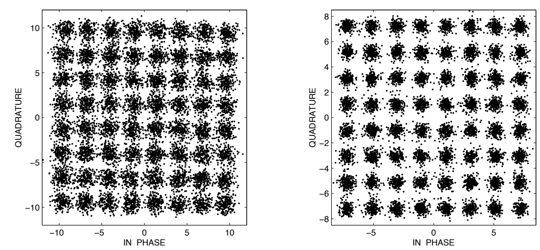


**Figure 5**. Result of equalization after blind identification. Traditional method [4] (left), and frequency domain approach of Sec. III (right).

In spite of its simplicity, the new method typically performs as well as the traditional method. Its main advantage is that the $L$ left-annihilators of the data matrix can be computed in closed form as in (15). Second, the channel coefficents $C(\rho_k)$ are also identified in closed form. The direct frequency-domain approach described above is therefore *inherently simple because of the closed-form solutions.* Since $P = M + L$ as in earlier methods (Sec. II), the precoder redundancy in the new method is the same as in the traditional method of Sec. II.

## V. REFERENCES


[1] Giannakis, G. B. "Filter banks for blind channel identification and equalization," IEEE Signal Processing Letters, vol. 4, no. 6, pp. 184–187, June 1997.

[2] Lin, Y.-P., and Phoong, S.-M., "Perfect discrete multitone modulation with optimal transceivers," IEEE Trans. SP, vol. 48, pp. 1702–1712, June. 2000.

[3] Scaglione, A., Giannakis, G. B., and Barbarossa, S. "Redundant filter bank precoders and equalizers Part I: Unification and optimal designs", IEEE Trans. Signal Processing, vol. 47, pp. 1988-2006, July 1999.

[4] — "Redundant filter bank precoders and equalizers Part II: Synchronization and direct equalization", IEEE Trans. Signal Processing, vol. 47, pp. 2007-2022, July 1999.

[5] Vaidyanathan, P. P. *Multirate systems and filter banks,* Prentice Hall, Inc., 1993.

[6] Vaidyanathan, P. P. and Vrcelj, B. "Transmultiplexers as precoders in modern digital communication: a tutorial review," IEEE ISCAS 2004, submitted.